

ICE-EM: MEASURE THEORY

PRELIMINARY NOTES ON ANALYSIS

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REFERENCES

Real Mathematical Analysis, C. Pugh, Springer, 2001.

Real Analysis, 3rd ed., H. Royden, Springer, 1988.

INTRODUCTION

Measure Theory is based upon the idea of taking a universal set X and considering the “size” of the subsets of X . The theory is extremely general: there are many choices for the world X , and of the notion of size or *measure* within X . The simplest and most natural choices for X are the set \mathbb{R} of real numbers and, more generally, the Euclidean Spaces, \mathbb{R}^m . These will be our main concern in the ICE-EM course. However, we will definitely consider measures in more general worlds as well.

No matter the choice of X , the most interesting measures are those whose properties are intimately related to the properties of X . We are then led to consider concepts of *analysis*, such as *distance* and *continuity* within X . Within \mathbb{R}^m , we are familiar with (and are about to be reminded of!) such concepts. More generally, we want to consider worlds where such concepts are defined, and this leads us to the worlds of *metric spaces* and *topological spaces* (see below).

These notes are a summary of the important (for us) definitions and theorems of analysis in Euclidean space, metric spaces and topological spaces. There are no proofs or examples. It is largely meant to be a summary of what you should already know, at least for Euclidean space. To the extent that the theorems and proofs are unfamiliar to you, you should (in an ideal world) try to prove the theorems, or (in the real world) choose a couple of appealing analysis texts and work through at least some of the proofs.

As the material is very standard, there are tons of texts to choose from, and *Pugh* and *Royden* are merely suggestions. From *Pugh*, selected parts of the first two chapters would be worth reading. *Royden* is pitched at a higher level, and is not the place to start if you’re unsure of your abilities: Chapters 2 and 7 are the most relevant.

You needn't make yourself an expert on 100% of the material here, especially the latter parts; we will cover and review the less familiar material as needed. But you should have a general sense of the material, and of the methods of proof involved. In brief, the Measure Theory course will be quite heavy with $\epsilon - \delta$ arguments, and it is expected that you will be able to understand and construct such proofs. Again, I won't expect you to enter the course as ϵ -experts. But, the more comfortable you are with the nature of analysis proofs, the better prepared you'll be.

THE REAL NUMBER SYSTEM

We take as given the sets \mathbb{N}, \mathbb{Z} and \mathbb{Q} of, respectively, *natural numbers*, *integers* and *rational numbers*; we merely note that \mathbb{N} is effectively characterized by *mathematical induction*: if $A \subset \mathbb{N}$ and

1. $1 \in A$,
2. $j \in A \implies j + 1 \in A$,

then $A = \mathbb{N}$. The constructions of \mathbb{Z} and \mathbb{Q} from \mathbb{N} are then quite natural and easy.¹

The big step of analysis is to fill in the gaps, to extend \mathbb{Q} to the set \mathbb{R} of *real numbers*. This can be done either *constructively* (where we somehow *make* the real numbers out of the rational numbers), or *axiomatically* (where we simply list certain fundamental properties which we are willing to accept as true). Here we'll do neither: we'll be content to recall the signature properties of \mathbb{R} , without regard to whether they are assumed or proved.²

The quickest (if not the easiest to digest) method of characterizing \mathbb{R} is with the concept of *least upper bound*. Given a set $A \subset \mathbb{R}$, we say A is *bounded above* if there is an *upper bound* M for A : that is, for every a ,

$$a \in A \implies a \leq M.$$

Then α is the *least upper bound* for A if:

1. α is an upper bound for A ;
2. β is an upper bound for $A \implies \alpha \leq \beta$.

Least Upper Bound Property. *If $A \subset \mathbb{R}$ is non-empty, and if A is bounded above, then A has a least upper bound α .* In this case, we write

$$\alpha = \sup A.$$

¹See, e.g., Part 1 of *Calculus* by M. Spivak, Publish or Perish, 3rd ed, 1994.

²An axiomatic system of the real numbers consists of *algebraic* laws, such as the *Commutative Law* ($a + b = b + a$), and laws of *ordering*, such as $a < b \implies a + c < b + c$. However the system \mathbb{Q} will also satisfy all such laws. What is needed is one last axiom, to capture the "no gaps" property of the real numbers. Any of the limit-type properties we now list could be taken as this final axiom.

Similarly, we can consider sets which are *bounded below*, and a non-empty set A which is bounded below will have a *greatest lower bound* $\beta = \inf A$.

An alternative characterization of \mathbb{R} can be made via *infinite sequences*. A real-valued sequence $\{a_j\}_{j=1}^{\infty}$ is *increasing* if $a_{j+1} \geq a_j$ for all $j \in \mathbb{N}$.³ $\{a_j\}$ *converges* to $a \in \mathbb{R}$, if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $j \geq N \implies |a_j - a| < \epsilon$. In this case, we write $a = \lim_{j \rightarrow \infty} a_j$ or $a_j \rightarrow a$. We now have:

Monotonic Sequence Property. *If $\{a_j\}$ is an increasing sequence of real numbers, and if $\{a_j\}$ is bounded above, then $\{a_j\}$ converges.*

Similarly, any *decreasing* sequence which is *bounded below* converges. We can also combine the previous concepts: if $\{a_j\}$ is bounded (above and below), but not necessarily monotonic, we can still define

$$\left\{ \begin{array}{l} \limsup_{j \rightarrow \infty} a_j = \lim_{j \rightarrow \infty} \sup_{k \geq j} a_k, \\ \liminf_{j \rightarrow \infty} a_j = \lim_{j \rightarrow \infty} \inf_{k \geq j} a_k. \end{array} \right.$$

We can also characterize \mathbb{R} in terms of *continuous functions*. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at a if, for every $\epsilon > 0$, there is a $\delta > 0$ such that, $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$. Then:

Intermediate Value Theorem. *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and if $f(a) < 0$ and $f(b) > 0$, then there is a $c \in (a, b)$ such that $f(c) = 0$.*

Further key properties of \mathbb{R} are implied in the results below. As a final property here, we note that, whereas \mathbb{Q} is *countable*, \mathbb{R} is *uncountable*: that is, there is no *bijection* (i.e. one-to-one correspondence) between \mathbb{N} and \mathbb{R} .

EUCLIDEAN SPACE

Given \mathbb{R} and $m \in \mathbb{N}$, we define *m-dimensional Euclidean Space*:

$$\mathbb{R}^m = \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \cdots, x_m) : x_j \in \mathbb{R} \text{ for } j = 1, \cdots, m\}.$$

Of course, choosing $m = 1$ includes \mathbb{R} as a special case. In \mathbb{R}^m we can define the *inner product*,

$$\langle x, y \rangle = \sum_{j=1}^m x_j y_j, \quad x, y \in \mathbb{R}^m,$$

and the associated *norm*,

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

³Technically, a real-valued sequence $\{a_j\}$ is a *function* $a : \mathbb{N} \rightarrow \mathbb{R}$: we're simply writing a_j for $a(j)$. Note also that an "increasing" sequence need not be *strictly* increasing.

We can then define

$$d(x, y) = \|x - y\|,$$

the *distance* between $x, y \in \mathbb{R}^m$. Then, as for \mathbb{R} , we can define the concepts of convergent sequences, and *Cauchy* sequences: a sequence $\{x_j\}$ in \mathbb{R}^m is Cauchy if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $j, k \geq N \implies \|x_j - x_k\| < \epsilon$. The simplest “fill the gaps” property of Euclidean space is then:

Completeness of Euclidean Space. *Every Cauchy sequence in \mathbb{R}^m converges.*

An equivalent characterization can be made in terms of the convergence of *infinite series*. Recall that for an infinite series $\sum_{j=1}^{\infty} x_j$ in \mathbb{R}^m we define the *sequence* $\{s_j\}$ of *partial sums*, where $s_j = \sum_{i=1}^j x_i$.

Then the series $\sum x_j$ is defined to converge if the sequence $\{s_j\}$ converges. A series $\sum_{j=1}^{\infty} x_j$ in \mathbb{R}^m

is said to *converge absolutely* if the series $\sum_{j=1}^{\infty} \|x_j\|$ (of real numbers) converges. We then have:

Absolute Convergence Test. *Every absolutely convergent series in \mathbb{R}^m converges.*

NORMED SPACES AND INNER PRODUCT SPACES

A real *vector space* X is a *normed space* if there is a function $\|\cdot\| : X \rightarrow \mathbb{R}$ with the following properties:

1. $\|x\| \geq 0$, and $\|x\| = 0$ iff $x = 0$;
2. $\|\alpha x\| = |\alpha| \cdot \|x\|$, for all $x \in X$ and $\alpha \in \mathbb{R}$;
3. $\|x + y\| \leq \|x\| + \|y\|$ (**Triangle Inequality**).

A real vector space X is an *inner product space* if there is a function $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{R}$ such that

1. $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ iff $x = 0$;
2. $\langle x, y \rangle = \langle y, x \rangle$;
3. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$.

Of course, \mathbb{R}^m is an inner product space; and, as for \mathbb{R}^m , any inner product is a normed space, by defining

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

We then have the **Cauchy-Schwartz Inequality**:

$$\langle x, y \rangle \leq \|x\| \cdot \|y\|.$$

METRIC SPACES

A *metric space* (X, d) is a set X together with a *distance function* $d : X \times X \rightarrow \mathbb{R}$ satisfying:

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$. (**Triangle Inequality**).

Any normed space, and \mathbb{R}^m in particular, is naturally a metric space by defining

$$d(x, y) = \|x - y\|.$$

We can then define convergent and Cauchy sequences as in Euclidean space. A metric space is said to be *complete* if every Cauchy sequence is convergent. A complete normed space is called a *Banach Space*, and a complete inner product space is called a *Hilbert Space*. In a normed space, we also have the notion of infinite series being convergent; then a normed space X is a Banach Space iff every absolutely convergent series is convergent.

Given a metric space X , $x \in X$ and $r \in \mathbb{R}$, we define, respectively, the *open ball* and *closed ball* of radius r about x :

$$\begin{cases} O_r(x) = \{y \in X : d(x, y) < r\}, \\ C_r(x) = \{y \in X : d(x, y) \leq r\}. \end{cases}$$

A set $O \subset X$ is *open* if for every $a \in O$ there is an $r > 0$ such that $O_r(x) \subset O$. A set $C \subset X$ is *closed* if its *complement* $X \sim C$ is open. Equivalently, C is closed if, for any sequence $\{x_j\}$ in C , whenever x_j converges to $x \in X$ then $x \in C$. Note that open balls are open and closed balls are closed, and that X and \emptyset are both open and closed.

If $A \subset X$ is non-empty and $x \in X$, the *distance* from x to A as

$$d(A, x) = \inf\{d(a, x) : a \in A\}.$$

Given $r \in \mathbb{R}$, $\{x \in X : d(A, x) < r\}$ is open and $\{x \in X : d(A, x) \leq r\}$ is closed. The *diameter* of a non-empty set A is defined by

$$\text{diam}(A) = \sup\{d(a, b) : a, b \in A\}$$

if this sup exists, and $\text{diam}(A) = \infty$ otherwise; A is *bounded* if $\text{diam}(A) < \infty$.

The *interior* A° of $A \subset X$ is the union of all open sets contained in A , and the *closure* \bar{A} of A is the intersection of all closed sets containing A . Note that \bar{A} is the union of A with any limit $x \in X$ of a sequence $\{x_j\}$ in A which converges. A is *dense* in X if $\bar{A} = X$; note that $\bar{\mathbb{Q}}$ is dense in \mathbb{R} and, more generally, $\bar{\mathbb{Q}^m}$ is dense in \mathbb{R}^m .

A metric space X is *compact* if any of the following equivalent conditions hold:

- Any sequence in X has a convergent *subsequence*,⁴ (**Bolzano-Weierstrass Property**);
- X is complete and *totally bounded*;⁵
- From any collection $\{O_\alpha\}$ of open sets which covers X , there is a finite subcollection which also covers X . (**Heine-Borel Property**).
- If $\{C_\alpha\}$ is a collection of closed sets in X such that any finite subcollection from $\{C_\alpha\}$ has non-empty intersection, then all of $\{C_\alpha\}$ has non-empty intersection.

Note, in particular, that a compact metric space is bounded.

A real-valued function $f : X \rightarrow \mathbb{R}$ on a metric space X is *continuous* if any of the following equivalent conditions hold:

- For any $x \in X$ and $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$;
- For any $\{x_j\}$ in X , if $x_j \rightarrow x$ then $f(x_j) \rightarrow f(x)$;
- For any open $O \subset \mathbb{R}$, the inverse image $f^{-1}(O)$ is open in X .

If $A \subset X$ then the function $x \mapsto d(A, x)$ is continuous. If $f : X \rightarrow \mathbb{R}$ is continuous and X is a compact metric space, then f is *uniformly continuous*: for any $\epsilon > 0$ there is a $\delta > 0$ such that, for all $x, y \in X$, $d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$.

TOPOLOGICAL SPACES

A *topological space* is a set X together with a collection $\mathcal{T} = \{O_\alpha\}$ of subsets of X , designated as the *open sets*. This collection must satisfy:

1. \emptyset and X are open;
2. If A and B are open then $A \cap B$ is open;
3. The union of any collection of open sets is open.

A *base* for the topology \mathcal{T} is a subcollection \mathcal{B} of open sets such that: for each $O \in \mathcal{T}$ and each $x \in O$ there is a $B \in \mathcal{B}$ such that $x \in B \subset O$. Any base completely determines the topology \mathcal{T} of X . For $x \in X$, a *base at x* is a collection \mathcal{B}_x of open sets such that: for each $O \in \mathcal{T}$ containing x there is a $B \in \mathcal{B}_x$ such that $x \in B \subset O$. If X is a metric space then the open balls are a base for X , and the open balls centred at a given x are a base at x .

⁴A subsequence $\{x_{j_k}\}_k$ of sequence $\{x_j\}_j$ is determined by a strictly increasing function (sequence) $k \mapsto j_k$ of natural numbers.

⁵ X is totally bounded if, for any $\epsilon > 0$, X is contained in finitely many open balls of radius $\epsilon > 0$.

Closedness, compactness and continuity in a topological space can be defined as for metric spaces above, where we restrict to those characterizations made in terms of open sets. In particular, if X and Y are topological spaces then $f : X \rightarrow Y$ is defined to be continuous if $f^{-1}(O)$ is open in X for every open $O \subset Y$. We note the following elementary properties:

- A closed subset of a compact topological space is compact;
- If $f : X \rightarrow Y$ is continuous and X is compact then $f(X)$ is compact.
- If $f : X \rightarrow \mathbb{R}$ is continuous and X is compact then f is bounded above (and below), and f achieves its maximum (and minimum) on X .
- If X is a topological space, and if $\{f_j : X \rightarrow \mathbb{R}\}$ is a sequence of continuous functions *converging uniformly* to f ,⁶ then f is continuous.

For a topological space X , define

$$C(X) = \{f : f \text{ is a continuous real-valued function on } X\}.$$

If X is a compact topological space, a consequence of the above results is that $C(X)$ is a normed space with the norm

$$\|f\| = \max\{f(x) : x \in X\}.$$

$C(X)$ is complete with the metric induced by this norm.

A topological space X is *locally compact* if for every $x \in X$ there is an open set O containing x with \overline{O} compact. Clearly, \mathbb{R}^m is locally compact.

If X and Y are topological spaces then the product $X \times Y$ can be given the *product topology*. A base for the product topology is the collection of all sets of the form $O_X \times O_Y$, where O_X is open in X and O_Y is open in Y . The product topology is the weakest topology (i.e. the topology including the least “open” sets) such that the *projections* $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ are continuous. If X and Y are compact (or locally compact) then so is $X \times Y$.

A topological space X is *metrizable* if there is a metric d on X which gives rise to exactly the open sets of X . Not all topological spaces are metrizable; and, if a topological space is metrizable, there will be many metrics which give rise to the same topology. If X and Y are metrizable, then so is $X \times Y$: if d_X metrizes X and d_Y metrizes Y , then

$$d((a, b), (c, d)) = d_X(a, c) + d_Y(b, d)$$

defines a metric which gives rise to the product topology on $X \times Y$.

Countability properties and *separation properties* indicate the extent to which a topological space behaves like a metric space. A topological space X is *separable* if it has a countable dense subset. X is *first countable* if there is a countable base at every $x \in X$, and X is *second countable* if X has a countable base. We note:

⁶ $\{f_j\}$ converges uniformly to f if, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $j \geq N \implies |f_j(x) - f(x)| < \epsilon$ for all $x \in X$.

- \mathbb{R}^m is separable;
- Any metric space is first countable;
- A metric space is second countable iff it is separable;

A topological space X is *Hausdorff* if, for any distinct points $c, d \in X$, there are disjoint open sets A and B with $c \in A$ and $d \in B$. A Hausdorff topological space is *normal* if, for any disjoint closed sets C and D , there are disjoint open sets A and B with $C \subset A$ and $D \subset B$. We note that:

- Any compact subset of a Hausdorff space is closed;
- Any singleton subset $\{a\}$ of a Hausdorff space is closed;
- Any metrizable topological space is normal;

Two deeper results are:

Urysohn's Lemma. *Suppose C and D are disjoint closed subsets of a normal space X . Then there is a continuous function $f : X \rightarrow \mathbb{R}, 0 \leq f \leq 1$, such that $f = 0$ on C and $f = 1$ on D .*

Urysohn Metrization Theorem. *If X is normal and second countable then X is metrizable.*

If X is locally compact and Hausdorff then Urysohn's Lemma still holds, as long as we assume one of the disjoint closed sets, C or D , is compact.