On Piecewise Fixed-Width Estimation of Process Capability Index: C_{pm}

A. Nanthakumar[1] and Basil M. de Silva[2]

[1]SUNY- Oswego, U.S.A and [2]RMIT University, Australia

Presenter: A. Nanthakumar, Department of Mathematics, SUNY- Oswego, Oswego, NY 13126, U.S.A. E-mail: nanthaku@oswego.edu

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Abstract

In this paper, we study the properties of the sequential fixed-width estimation of the quality index C_{pm} in a piecewise manner. We show that this piecewise fixed-width rule terminates finitely when the fixed-width 2d > 0. Under some reasonable conditions, we show that the fixed width estimation is asymptotically efficient and asymptotically consistent, as d tends to zero.

1. Introduction

To control the quality, there are several quality control methods available in the industry and some of these methods involve the use of the process capability indices such as C_p, C_{pk} and C_{pm} to monitor a single process characteristic. Sullivan (1984), Kane (1986), Chan, Cheng and Spiring (1988) and Gunter (1989) discuss the use of the process capability indices in quality control. Primarily, there are two kinds of indices. The first kind deals with the process variation and the second kind deals with the deviation of the process mean from the target value. For example, the process capability index C_p belongs to the first kind and C_{pk}, C_{pm} belong to the second kind. Also, note that C_{pk} and C_{pm} incorporate the process variation measure and therefore have an advantage in estimating the quality. Chan, Chen and Spring (1988) and Subbaiah and Taam (1993) prefer the use of C_{pm} to C_{pk} as the exact sampling distribution for the estimate of C_{pk} is more complex than the one for C_{pm}. It should be noted that the estimates for the process mean and the process variation depend on the sample size. So, in the case of fixed samples, the choice for the sample size may play a part in the decision whether or not to accept a product on the basis of the estimate for C_{pm}. In order to circumvent this possible dependence on the fixed sample size, we use the sequential sampling, and this allows a stopping rule to decide the necessary sample size. Wald (1947) was the first to suggest the use of stopping rules in the sample size selection. Since then, several authors, Anscombe (1953), Robbins (1965), Starr (1966), Lai and Siegmund (1977), Woodroofe (1982) and many others have done extensive studies on sampling. Interested readers are referred to the excellent book by Govindarajulu (1981) for the literature review on sequential sampling mthods. Recently, Mukhopadhyay and Sen (1993), Mukhopadhyay and Datta (1994), Mukhopadhyay and de Silva (2002) have used the idea of a piecewise sequential methodology where a "large" purely sequential experiment is implemented in several pieces thereby achieving significant operational convenience exploiting the basic ideas of parallel processing.

In this paper, the process capability index C_{pm} is estimated by using this piecewise sequential methodology. Nanthakumar and Selvavel (2003) discuss the estimation of C_{pm} by using a purely sequential fixed-width rule. Here, we discuss the estimation of C_{pm} by using several fixed-width rules in a piecewise manner. In an industrial setting, this piecewise estimation is more appealing and convenient and leads to speedier decisions concerning quality. Towards this piecewise sequential estimation of C_{pm}, we derive a fixed-width stopping rule, and then on the basis of several of these stopping rules, we collect the measurements at random in a parallel manner. This paper deals with the theoretical merits of this piecewise sequential estimation. We structure our paper in a way that Section 2 contains the relevant methodology; Section 3 contains the main results and Section 4, the numerical results. We present the results in the main body of the paper. For the interested readers, we attach the proofs in the appendix section.

2. Methodology

Consider a quality characteristic X following a distribution F, with L and U representing the lower and the upper specification limits for the acceptable quality characteristic and T = (L+U)/2 be the target value for the production process. The process capability index C_p is defined as $C_p = (U-L)/6\sigma$ is the ratio of the allowable process spread for the variation and the actual process spread for the variation. There are several process capability indices. For example, the process capability index $C_{pk} = \min(U-\mu, \mu-L)/3\sigma$ where μ and σ are the mean and the standard deviation for this quality characteristic. Marucci and Beazley (1988) introduced another process capability index

$$C_{pm} = (U-L)/6\sigma_T .$$

Note that $\sigma_T^{\,2} = E(X-T)^2 = \sigma^2 + (\mu-T)^2$. Also, one can write

$$C_{pm} = \frac{(U-L)}{6\sigma}\left\{1+\frac{(\mu-T)^2}{\sigma^2}\right\}^{-1/2}$$

$$= C_p/D, \qquad (2.1)$$

where $D = \left\{1+\frac{(\mu-T)^2}{\sigma^2}\right\}^{1/2}$ and this D measures the deviation of the process mean from the target. Moreover, C_{pm} incorporates the process deviation from the target. Therefore, in this paper, we focus on C_{pm} to measure the quality. The estimate of C_{pm} is given by

$$\hat{C}_{pm} = \frac{(U-L)}{6s_{n-1}}\left\{1+\frac{n}{(n-1)}\frac{\left(\overline{X}_n - T\right)^2}{s_{n-1}^2}\right\}^{-1/2} \qquad (2.2)$$

(see Subbaiah and Taam (1993) for details), where $\overline{X}_n$ and s_{n-1}^2 are the sample mean and the sample variance respectively. Marcucci and Beazley (1988) and Boyles (1991) defined an alternative estimate for C_{pm}, and it is given by

$$\tilde{C}_{pm} = \sqrt{\frac{n}{(n-1)}}\ \hat{C}_{pm}. \tag{2.3}$$

As stated in Subbaiah and Taam (1993), the mean square error for $\hat{C}_{pm}$ is less than the mean square error for $\tilde{C}_{pm}$. Therefore, we use $\hat{C}_{pm}$ in the sequential estimation of the process capability index C_{pm}.

The following lemma and corollary show the connection between the quality measure C_{pm} and the probability for product quality acceptance.

Lemma 1: If the quality characteristic X follow a distribution with cdf F(.), then

$$P(L \le X \le U) = F(T + 3\sigma_T C_{pm}) - F(T - 3\sigma_T C_{pm}) \tag{2.4}$$

where $T = (L + U)/2$ and $\sigma_T^2 = E(X-T)^2$.

Corollary: If $\mu = T$, then

$$P(L \le X \le U) = F(3C_{pm}) - F(-3C_{pm}). \tag{2.5}$$

In process control, the process is determined to be in control if the quality characteristic is between the threshold limits L and U. As seen from Lemma 1 and the follow up corollory, the quality index C_{pm} is a good measure of the quality. This is another reason for estimating C_{pm} in the context of process quality control.

Next, we state the results that we need in order to derive the fixed-width stopping rule and the piecewise stopping rule to estimate C_{pm} sequentially.

Result 2:

For the estimate $\hat{C}_{pm}$ given in (2.2)

(i) $E\left(\hat{C}_{pm}\right) = \sqrt{\frac{(n-1)}{n}}\left(1+\frac{3}{4\upsilon}\right)C_{pm} + O\left(n^{-2}\right)$,

(ii) $Var\left(\hat{C}_{pm}\right) = \frac{(n-1)}{2n\upsilon} C_{pm}^2 + O\left(n^{-2}\right)$,

(iii) $\sqrt{n}\ \left(\hat{C}_{pm} - C_{pm}\right) \xrightarrow{L}$ N(0, λ),

$$\text{where } \lambda = \frac{\sigma^2}{2}\frac{\left(\sigma^2+2(\mu-T)^2\right)}{\left(\sigma^2+(\mu-T)^2\right)^2}C_{pm}^2 \quad \text{and } \nu = \frac{n}{\sigma^2}\frac{\left(\sigma^2+(\mu-T)^2\right)^2}{\left(\sigma^2+2(\mu-T)^2\right)}.$$

Parts (i) and (ii) follow from Subbaiah and Taam (1993) and part (iii) follows from Chan et al (1990). Note that part (iii) is used in the derivation of the stopping rule.

Derivation of the Piecewise Sequential Rule

First, we derive the "purely" sequential stopping rule for the sample size selection such that stop taking the (sample) observations immediately when the probability that the absolute value deviation between the estimate of C_{pm} and C_{pm} being less than d (say) is at least 1 - α, where 0 < α < 1. That is, stop taking observations when

$$P\left\{\left|\hat{C}_{pm} - C_{pm}\right| \le d\right\} \ge 1-\alpha .$$

That is equivalent to stop taking observations when $\sigma_{\hat{C}_{pm}}$ is the standard error of $\hat{C}_{pm}$ and $Z_{\alpha/2}$ is the (1 - α /2)th quantile for the standard normal distribution. So, when the parameters are known, follow the "purely" sequential stopping rule

$$N_o = \inf\left\{n : n \ge \frac{\lambda}{d^2} Z^2{}_{\alpha/2}\right\} \tag{2.6}$$

and when the parameter λ is unknown, follow the "purely" adaptive rule

$$N_1 = \inf\left\{ n : n \geq \frac{\hat{\lambda}}{d^2} Z_{\frac{\alpha}{2}}{}^2 \right\}, \tag{2.7}$$

where $\hat{\lambda}$ is the estimate of λ.

However, we are dealing with piecewise sequential rules, and the "piecewise" sequential rules are given as follows.

$$N_{0,i} = \inf \{ n: n \geq \frac{\lambda}{\rho} (z_{\alpha/2} / d)^2 \} \quad i = 1,\text{------}, R \tag{2.8}$$

where ρ is an "accelerator factor" and when λ is unknown, we can use the rule given below.

$$N_{1,i} = \inf \{ n: n \geq \frac{\hat{\lambda}}{\rho} (z_{\alpha/2} / d)^2 \} \quad i = 1,\text{------},R \tag{2.9}$$

where R is the number of subgroups that are involved in this process control operation.

Define

$$M_0{}^1 = \sum_{i=1}^{R} N_{0,i} \tag{2.10}$$

$$M_1{}^1 = \sum_{i=1}^{R} N_{1,i} \tag{2.11}$$

Define $M_1 = \left[\frac{\hat{\lambda}_{M_1}{}^1}{d^2} z_{\alpha/2}{}^2 \right] + 1$ and $M_0 = \left[\frac{\lambda}{d^2} z^2{}_{\alpha/2} \right] + 1$

and let $N = \text{Max}(M_1 , M_1{}^1)$ and $N_0 = \text{Max}(M_0 , M_0{}^1)$.

Note that N is the required number of observations when the parameter λ is unknown. Similarly, N_0 is the required number of observations for the estimation when the parameter λ is known.

Next, we present the theoretical and numerical results separately. First, we present the theoretical results.

3. Results

Lemma 1:

$\mathbf{N} \rightarrow \infty$ as d $\rightarrow 0$.

Lemma 2:

The total stopping value terminates finitely with probability one when d > 0.

Lemma 3:

(i). $\frac{N}{N_0} \xrightarrow{as} \frac{R}{\rho}$ or $\frac{\rho}{R}$ as d $\longrightarrow 0$

(ii). If $E(s_n^{-8}\sigma^8) = 1 + O(n^{-\alpha})$ for $\alpha \geq 1$, then

$$E\left(\frac{N}{N_0}\right) \rightarrow \frac{\rho}{R} + \left(1 - \frac{\rho}{R}\right)\delta \text{ as d} \longrightarrow 0 \quad \text{if } M_0^{\ 1} > M_0$$

$$\rightarrow 1 + \left(\frac{R}{\rho} - 1\right)\delta \text{ as d} \rightarrow 0 \quad \text{if } M_0^{\ 1} \leq M_0$$

where δ is the probability that $M_1^{\ 1} > M_1$.

Note: Lemma 3 establishes the asymptotic efficiency when ρ = R. Also note that when ρ > R and $M_0{}^1 \le M_0$ or when ρ < R and $M_0{}^1 \ge M_0$ then the limit of $E\left(\frac{N}{N_0}\right) < 1$ as $d \to 0$. Similarly, when ρ < R and $M_0{}^1 \le M_0$ or when ρ > R and $M_0{}^1 \ge M_0$ then the limit of $E\left(\frac{N}{N_0}\right) > 1$ as $d \to 0$. However, the chance for the second scenario is very small and therefore in most cases, the limit of $E\left(\frac{N}{N_0}\right) \le 1$ as $d \to 0$. Our simulation studies also seem to confirm this finding.

Lemma 4:

Let $\hat{C}_{pm,N}$ and $\hat{C}_{pm,N_0}$ be the estimates of C_{pm} based on the total stopping values given by the piecewise stopping rules. If R = ρ , where ρ is the accelerator factor then

(i) $$\frac{C_{pm,N} - C_{pm,N_0}}{\sigma_{\hat{C}_{pm,N_0}}} \to N(0,1) \text{ as } d \to 0.$$

(ii) $\to 1$ as $d \to 0$.

Note that the above result establishes the asymptotic consistency of the stopping rule.

Numerical Results.

Here, we present the numerical results based on 100 simulated replications. The statistical software SAS was used for simulating the replications. Note that the nominal coverage probability is 0.95 .

d	R	Estimate of E(N)	N_0	Coverage Probability Estimate
0.05	5	149.6	158	0.92
0.05	10	166.6	158	0.93
0.07	5	74.7	81	0.90
0.07	10	78.9	81	0.91

2d = Width of the fixed-width (nominal coverage) interval

R = Number of piecewise stopping rules

N = Total number of observations required when the parameter λ is unknown

N_0 = Total number of observations required when the parameter λ is known

Appendix

Lemma A.1:

For the sample mean $\overline{X}_n$ and the sample standard deviation s_n

(i) $\overline{X}_n \xrightarrow{as} \mu$ as n $\to \infty$.

(ii) $s_n \xrightarrow{as} \sigma$ as n $\to \infty$.

Proof:

The result follows from the strong law of large numbers.

Lemma A.2:

For the stopping rule, the mean and the standard deviation

(i) $\overline{X}_N \xrightarrow{as} \mu$ as d $\to 0$

(ii) $s_N \xrightarrow{as} \sigma$ as d $\to 0$

Proof:

The result follows as a result of Csorgo's theorem and the strong law of large numbers.

Csorgo's Theorem:

Let $\{Z_n\}$ be a sequence of random variables such that Z_n tends to a non-stochastic constant α as n tends to ∞. Let $\{v_n\}$ be a sequence of positive integer valued random variables.

(i) If $v_n \to \infty$ in probability then $Z_{v_n} \to \alpha$ in probability.

(ii) If $v_n \to \infty$ almost surely then $Z_{v_n} \to \alpha$ almost surely.

Next, we present the proofs for the lemmas in the main body of the paper.

Proof of Lemma 2:

For fixed d > 0, we can show that $\overline{X}_{N_{1,i}-1}$ and $s^2_{N_{1,i}-1}$ are finite with probability one. Hence, $\hat{\lambda}_{N_{1,i}-1}$ is finite with probability one due to the fact $\hat{\lambda}_{N_{1,i}-1}$ is a well-defined function of $\overline{X}_{N_{1,i}-1}$ and $s^2_{N_{1,i}-1}$. So, the right hand side of the above inequality is finite with probability one and therefore, the stopping time is finite with probability one for $d > 0$.

Proof of Lemma 3:

We will discuss this proof in four different cases.

Case 1: $M_0^{\,1} > M_0$ and $M_1^{\,1} > M_1$

$$\frac{N}{N_0} = \frac{\sum_1^R N_i^{\,1}}{\sum_1^R N_{0,i}^{\quad 1}} = \frac{1}{R}\sum_1^R \frac{N_i^{\,1}}{N_{0,i}^{\,1}} \to 1 \text{ almost surely as d } \to 0.$$

Case 2: $M_0^{\,1} > M_0$ and $M_1^{\,1} < M_1$

$$\frac{N}{N_0} = \frac{M_1}{N_0} = \frac{\left[\frac{\hat{\lambda}_{M_1}}{d^2} z^2_{\alpha/2}\right]+1}{\sum_1^R N_{0,i}^{\quad 1}} \to \frac{\rho}{R} \text{ almost surely as d } \to 0.$$

Case 3 : : $M_0{}^1 < M_0$ and $M_1{}^1 > M_1$

$$\frac{N}{N_0} = \frac{\sum_1^R N_i{}^1}{\left[\frac{\lambda}{d^2} z^2{}_{\alpha/2}\right]+1} \rightarrow \frac{R}{\rho} \text{ almost surely as d } \rightarrow 0.$$

Case 4: $M_0{}^1 < M_0$ and $M_1{}^1 < M_1$

$$\frac{N}{N_0} = \frac{\left[\frac{\hat{\lambda}_{M_1{}^1}}{d^2} z^2{}_{\alpha/2}\right]+1}{\left[\frac{\lambda}{d^2} z^2{}_{\alpha/2}\right]+1} \rightarrow 1 \text{ almost surely as d } \rightarrow 0.$$

To prove part (ii) of Lemma 3, consider the following cases,

Case 1. $M_1{}^1 > M_1$

We can easily show that $\frac{N}{N_0} \leq \sum_1^R \frac{N_i{}^1}{\left(\frac{\lambda z^2{}_{\alpha/2}}{d^2}\right)}$

and can rewrite this inequality as

$$\frac{N}{N_0} \leq \frac{(U-L)^2}{72\lambda\rho} \sum_1^R \left(s^2{}_{N_i-1} + 2\left(\overline{X}_{N_i-1} - T\right)^2\right) + \frac{Rd^2}{\lambda z^2{}_{\alpha/2}}$$

Case 2:   . $M_1{}^1 < M_1$

$$\frac{N}{N_0} \leq \frac{(U-L)^2}{72\lambda} \sum_{i=1}^{R} \left(s_{N_{1,i}-1}^{-4} \left(s_{N_{1,i}-1}^{2} + 2\left(\overline{X}_{N_{1,i}-1} - T\right)^2 \right) \right) + \frac{d^2}{\lambda z_{\alpha/2}^2}$$

However, if $E(s_n^{-8}\sigma^8) = 1 + O(n^{-\alpha})$ for $\alpha \geq 1$, then by Cauchy-Schwarz inequality, we can show that for the right hand side of the above inequality, its expected value is bounded by a finite number. Hence, by dominated convergence theorem, the result follows.

Note: If the quality characteristic X follows a normal distribution, then we can easily show that $E(s_n^{-8}\sigma^8) = 1 + 24/n + \ldots$ and this means that $E(s_n^{-8}\sigma^8) = 1 + O(n^{-1})$ and hence $\alpha = 1$.

Proof of Lemma 4:

The result follows from Anscombe's (1952) theorem and the theorem of Chan, Xiong and Zhang (1990) in addition to the fact $\frac{N}{N_0}$ tends to 1 as d tends to 0.

Proof of Lemma 4 (i):

Note that

$$P\left(\left| \hat{C}_{pm,N} - C_{pm} \right| < d \right)$$

$$= P\left(\left| \frac{\hat{C}_{pm,N} - C_{pm}}{\sigma_{\hat{C}_{pm,N_0}}} \right| < \frac{d}{\sigma_{\hat{C}_{pm,N_0}}} \right)$$

$$\rightarrow \Phi(Z_{\alpha/2}) - \Phi(-Z_{\alpha/2}) \text{ as } d \rightarrow 0$$

$$\rightarrow 1 - \alpha/2 - \alpha/2 = 1 - \alpha .$$

The reason is that $\frac{d}{\sigma_{\hat{C}_{pm,M_0}}} \rightarrow z_{\alpha/2}$ as $d \rightarrow 0$ and by Lemma 4, $\frac{(\hat{C}_{pm,M_1} - \hat{C}_{pm,M_0})}{\sigma_{\hat{C}_{pm,M_0}}}$

follows a standard normal distribution. Also, note that

$$\mathrm{P}\left(\left|\hat{C}_{pm,M_0} - C_{pm}\right| < d\right) = \mathrm{P}\left(\left|\frac{\hat{C}_{pm,M_0} - C_{pm}}{\sigma_{\hat{C}_{pm,M_0}}}\right| < \frac{d}{\sigma_{\hat{C}_{pm,M_0}}}\right)$$

$$\rightarrow \quad \Phi(Z_{\alpha/2}) - \Phi(-Z_{\alpha/2})$$

$$\rightarrow \quad (1 - \alpha) \quad \text{as } d \rightarrow 0.$$

Hence the proof.

References:

1. Anscombe, F. J. (1953). Sequential Estimation. *Journal of Royal Statististical Society Series B*, 15, 1-29.
2. Boyles, R. A. (1991). The Taguchi Capability Index, *Journal of Quality Technology*, 23, 1, 17-26.
3. Chan, L. K., Cheng, S. W. and Spiring, F. A. (1988). A New Measure of Process Capability: C_{pm}, *Journal of Quality Technology*, 20, 3, 162-175.
4. Chan, L. K., Xiong, Z., and Zang, D. (1990). On the asymptotic distribution of some process capability indices. *Communications in Statistics - Theory and Methods*, 19, 1, 11-18.
5. Chow, Y. S. and Robbins, H. (1965). On the asymptotic Theory of Fixed-width Sequential Confidence Intervals for the Mean. *Annals of Mathematical Statististics*, 36, 457-462.
6. Csorgo, M. (1968). On the strong law of large numbers and the central limit theorem for martingales. *Transactions. American Mathematical Society*, 131 (1968), 259-275.
7. Govindarajulu, Z. (1981). The Sequential Statistical Analysis of Hypothesis Testing, Point and Interval Estimation, and Decision Theory. *American Sciences Press*. Columbus.

8. Gunter, B. H. (1989). The Use and Abuse of C_{pk}, *Quality Progress*, January, 72-73; March, 108-109; May, 79-80; July, 86-87.
9. Kane, V. (1986). Process Capability Indices, *Journal of Quality Technology*, 18, 1, 41-52.
10. Kocherlakota, S., Kocherlakota, K. and Kirmani, S. N. U. A. (1992). Process Capability Index under Non-normality, *International Journal of Mathematical and Statistical Science*, 1, 175-209.
11. Kushler, R. and Hurley, P. (1992). Confidence Bounds for Capability Indices, *Journal of Quality Technology*, 24, 188-195: Correction Vol. 25, p. 333.
12. Lai, T. L. and Siegmund, D. O. (1977). A Nonlinear Renewal Theory with Applications to Sequential Analysis. *Annals of Statististics*, 5, 946-954.
13. Marcucci, M. O. and Beazley, C. C. (1988). Capability Indices: Process Performance Measures, *ASQC Quality Congress Transactions*, 516-523.
14. Mukhopadhyay, N. and Datta, S. (1994). Replicated Piecewise Multistage Sampling with Applications. *Sequential Analysis*, 13(3), 253-276.
15. Mukhopadhyay, N. and Sen, P.K.(1993). Replicated Piecewise Stopping Numbers and Sequential Analysis, *Sequential Analysis*, 12(2), 179-197.
16. Mukhopadhyay, N.and de Silva, B.M.(1998). Multistage Partial Piecewise Sampling and its applications. *Sequential Analysis*, 17(1), 63-90.
17. Nanthakumar, A. and Selvavel, K.(2003). On Fixed-Width Estimation of Process Capability Index C_{pm} , *American Journal of Mathematical and Management Science* 23(1,2), 93-108.
18. Robbins, H. (1959). Sequential estimation of the mean of a normal population. Probability and Statistics (Harold Cramer Volume), 235-245.*Almquist and Wiksell*, Uppsala, Sweden.
19. Starr, N. (1966). On the performance of a sequential procedure for the fixed-width interval estimation of the mean, *Annals of Mathematical Statististics*, 37, 36-50.

20. Subbaiah, P. and Taam, W. (1993). Inference on the Capability Index: C_{pm}, *Communications in Statistics-Theory and Methods*, 22(2), 537-560.

21. Sullivan, L. P. (1984). Reducing Variability: A New Approach to Quality, *Quality Progress*, 17, 15-21.

22. Wald, A. (1947). Sequential Analysis. *Wiley*, New York.

23. Woodroofe, M. (1982). Nonlinear Renewal Theory in Sequential Analysis. *Society for Industrial and Applied Mathematics*. Philadelphia.